AD-A046 449 PURDUE UNIV LAFAYETTE IND DEPT OF MATHEMATICS F/G 12/1

STOCHASTICALLY PERTURBED LIMIT CYCLES, (U)

AUG 77 C J HOLLAND AFOSR-77-3296

UNCLASSIFIED AFOSR-TR-77-1276 NL

1 OF 1

ADA

046449

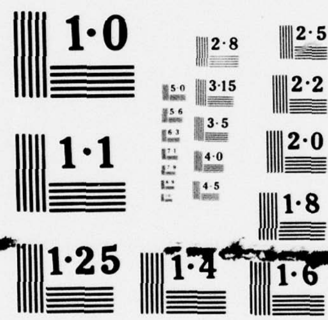
END

DATE

FILMED

12-77

DDC



NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

AFOSR-TR- 77-1276

Stochastically Perturbed Limit Cycles

(submitted to J. Applied Probability)

Charles J. Holland
Purdue University

August 1977

Accompanying Statement

An important question in the stability and control of stochastic systems is the determination of the limiting long time behavior of the system using a fixed control. This paper answers that question in the case of a stochastic system perturbed by a small additive noise term where the control is such that the corresponding deterministic system has a stable limit cycle.

It is shown that in the limit of large time the stochastic system is near the limit cycle. This is a stability result. Moreover, one can compute approximately at which portion of the limit cycle one is most likely to be found. Further various stationary averages can be computed.

These results will be of use in designing approximate controls for stationary stochastic control systems. As a by-product, the above work allows one to deduce some additional results on singular perturbation problems in partial differential equations.

This research was supported in part by AFOSR Grant 77-3296.

(See 1473)

Stochastically Perturbed Limit Cycles

Charles J. Holland

1. Introduction. In this paper we examine the effects of perturbing certain deterministic dynamical systems possessing a stable limit cycle by an additive white noise term with small intensity. We place assumptions on the system guaranteeing that when noise is present the corresponding random process generates an ergodic probability measure. We then determine the behavior of the invariant measure when the noise intensity is small. This can be considered as a purely probabilistic result. Using this probabilistic result we are also able to determine the limiting large time behavior of certain singularly perturbed second order partial differential equations associated with the random process through Itô's rule.

For a model of our probabilistic process we consider the Itô stochastic differential equation

$$d\xi = f(\xi)dt + (2\varepsilon)^{1/2}\sigma(\xi)dw(t) \ . \qquad (1)$$

In (1) f is an n vector, σ is an $n \times k$ matrix, w is k dimensional Brownian motion and ε is our small positive parameter. For $\varepsilon = 0$ we have the corresponding deterministic dynamical system. For $\varepsilon \geq 0$ denote by $\xi_x^\varepsilon(t)$ the solution to (1) with initial condition $\xi_x^\varepsilon(0) = x$. Then throughout the following assumptions are made.

(A1) f_i, σ_{ij} are of class $C^2(R^n)$.

(A2) For $\varepsilon = 0$ the system has a unique limit cycle denoted by Γ.

(A3) There exists at most a finite number of critical points (places x^* where $f(x^*) = 0$). At each critical point the matrix $f_x(x^*)$ has only eigenvalues with positive real parts. At each critical point the matrix $\sigma(x^*)\sigma^T(x^*)$ is positive definite.

(A4) For any compact set B not containing critical points and any $\delta > 0$ there exists $T > 0$ (depending upon B, δ) such that if $x \in B$, then

$$d(\xi_x^0(t),\Gamma) < \delta \quad \text{for} \quad t \geq T .$$

Here d denotes the Euclidean distance function.

(A5) There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the stochastic differential equation possesses an ergodic measure μ_ε with a density. For any $\delta > 0$ there exists $R > 0$ such that that

$$\mu_\varepsilon\{B(R)^c\} < \delta \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0$$

where $B(R) = \{x: |x| \leq R\}$.

Then we prove the following result.

<u>Theorem 1</u>. (C1). If B is any open set such that $\Gamma \subset B$, then

$$\lim_{\varepsilon \downarrow 0} \mu^\varepsilon(B) = 1 .$$

(C2). If H is continuous and bounded, then

$$\lim_{\varepsilon\downarrow 0} \int_{R^n} H(x)d\mu_\varepsilon(x) = \frac{1}{T^*}\int_0^{T^*} H(\xi_{\overline{x}}^0(t))dt \quad . \tag{2}$$

where $\overline{x}$ is any point on Γ and T^* is the period of the cycle, $\xi_{\overline{x}}^0(t) = \xi_{\overline{x}}^0(t+T^*)$.

We prove Theorem in Section 2 and discuss assumption (A5) there. From Theorem 1 we can immediately derive a result in partial differential equations under the following additional assumption.

(A6) There exists $\overline{\varepsilon} > 0$ such that for $0 < \varepsilon < \overline{\varepsilon}$ there is a bounded solution u^ε on $(t>0)\times R^n$ to

$$\text{trace }(\varepsilon\sigma\sigma^T u_{xx}) + fu_x - u_t = 0 \tag{3}$$

with initial condition $u^\varepsilon(0,x) = H(x)$, $x \in R^n$. The function H is assumed continuous and bounded as in Theorem 1.

Assumption (A6) is discussed in Section 2.

The solution u^ε to (3) satisfies $u^\varepsilon(t,x) = EH(\xi_x^\varepsilon(t))$. Since (1) has for each ε an ergodic measure, then, for all x

$$\lim_{t\to\infty} u^\varepsilon(t,x) = \int_{R^n} H(y)d\mu_\varepsilon(y) \quad . \tag{4}$$

Hence we have the following

Theorem 2. Assume (A1) - (A6). Then

$$\lim_{\varepsilon \to 0} \lim_{t \to \infty} u^{\varepsilon}(t,x) = \frac{1}{T^*} \int_0^{T^*} H(\xi_{\bar{x}}^{0}(t))dt \ . \tag{5}$$

However, for any $t > 0$, no matter how large, we have

$$\lim_{\varepsilon \to 0} u^{\varepsilon}(t,x) = H(\xi_x^0(t)) \ . \tag{6}$$

See Fleming [1], Theorem 4.1. Equations (5) and (6) show that caution must be observed when attempting to approximate the long term behavior of a physical system subject to small random disturbances.

In Section 3 we consider a class of problems for which both the invariant measure μ_ε and the limit cycle Γ can be computed explicitly. We evaluate directly both sides of equation (2).

In earlier work ([6],[7]) we have considered equation (1) when the corresponding deterministic dynamical system has the origin as a globally asymptotically stable equilibrium point. In Section 4 we consider that case and prove easily the following

Theorem 3. Assume (A1), (A5), and

(A2'). For $\varepsilon = 0$, (1) has a unique critical point at $x^* = 0$ which is globally asymptotically stable.

Then, if B is any open set containing the origin

$$\lim_{\varepsilon \downarrow 0} \mu_\varepsilon(B) = 1 \ . \tag{7}$$

If H is continuous and bounded, then

$$\lim_{\varepsilon \downarrow 0} \int_{R^n} H(x) d\mu_\varepsilon(x) = H(0) . \tag{8}$$

Under stronger assumptions and using a completely different technique we established in [6] an expansion of the left hand side of (8) in powers of ε. One might attempt to derive such a result for the left hand side of (2). We give an example to illustrate the potential difficulty in deriving such an expansion without strong assumptions.

2. In this section we first prove Theorem 1 and then discuss assumptions (A5) and (A6).

Proof of Theorem 1. We first establish (C1). To do this we begin by constructing for each critical point x^* a neighborhood N containing x^* such that

$$\lim_{\varepsilon \to 0} \mu_\varepsilon(N) = 0 . \tag{9}$$

For notational convenience let $x^* = 0$ denote a critical point.

Since assumption (A3) holds, there exists a positive definite matrix P, a positive constant c and a neighborhood Z of $x^* = 0$ such that if $V(x) = x^T Px$, then $V_x f \geq 0$ in $\overline{Z}$ (see [5], p. 296) and also trace $\sigma\sigma^T P \geq c$ in $\overline{Z}$. Now let c_1, c_2, M be positive constants with $c_1 > c_2 > M$ and such that the set $S = \{x : V(x) \leq c_1\} \subset Z$. Define $Q = \{x : V(x) < c_2\}$, $N = \{x : V(x) < M\}$, and let $0 < \gamma < 1/2$ be such that $d(\partial S, \partial Q) > 2\gamma$ and $d(\partial N, \partial Q) > 2\gamma$. We shall need these facts below.

We claim that (9) holds for the above N. Suppose not, then we shall show (in a lengthy argument) that this leads to a contradiction. First, since (9) is assumed false, there exists a positive number δ such that $\overline{\lim_{\varepsilon \to 0}} \mu_\varepsilon(N) > \delta$. Using assumptions (A5), (A3) we next choose R such that $\Gamma \subset B(R)$, all critical points lie in $B(R-1)$, and $\mu_\varepsilon(B(R)^c) < \delta/6$ for $0 < \varepsilon < \varepsilon_0$ for some fixed $\varepsilon_0 > 0$.

For each $\varepsilon > 0$, let $\tilde{\xi}^\varepsilon(t)$ be the solution to (1) with initial condition distributed according to the measure μ_ε. Define sets K_1^ε, K_2^ε, K_3^ε, K_4^ε by

$$K_1^\varepsilon = \{\tilde{\xi}^\varepsilon(\tfrac{2M}{c\varepsilon}) \in N, \quad \tau^\varepsilon \geq \tfrac{2M}{c\varepsilon}, \quad \tilde{\xi}^\varepsilon(0) \in N\} ,$$

$$K_2^\varepsilon = \{\tilde{\xi}^\varepsilon(\tfrac{2M}{c\varepsilon}) \in N, \quad \tau^\varepsilon < \tfrac{2M}{c\varepsilon}, \quad \tilde{\xi}^\varepsilon(0) \in N\} ,$$

$$K_3^\varepsilon = \{\tilde{\xi}^\varepsilon(\tfrac{2M}{c\varepsilon}) \in N, \quad \tilde{\xi}^\varepsilon(0) \in B(R) - N\} ,$$

$$K_4^\varepsilon = \{\tilde{\xi}^\varepsilon(\tfrac{2M}{c\varepsilon}) \in N, \quad \tilde{\xi}^\varepsilon(0) \in B(R)^c\} ,$$

where M is the maximum of V in $\overline{N}$ and τ^ε is the exit time of $\tilde{\xi}^\varepsilon(t)$ from N if $\tilde{\xi}^\varepsilon(0) \in N$.

Since $\mu_\varepsilon(N) = P\{\tilde{\xi}^\varepsilon(t) \in N\}$ for any $t > 0$, then

$$\mu_\varepsilon(N) = \sum_{i=1}^{4} P(K_i^\varepsilon) . \tag{10}$$

We need to estimate the terms on the right side of the previous equality. First, we have $P\{K_4^\varepsilon\} \leq \delta/6$.

Let us estimate $P\{K_1^\varepsilon\}$ next. If $x \in N$, let τ_x^ε be the exit time of $\xi_x^\varepsilon(t)$ from N. Then using the Itô stochastic differential rule we have for any $t > 0$ that

$$EV(\xi_x^\varepsilon(t \wedge \tau_x^\varepsilon)) - EV(x)$$

$$\geq E\int_0^{t\wedge\tau_x^\varepsilon} (V_x f + \varepsilon \text{ trace } P\sigma\sigma^T)(\xi_x^\varepsilon(t))dt \geq \varepsilon cE(t \wedge \tau_x^\varepsilon) .$$

Then $M \geq \varepsilon cE(t \wedge \tau_x^\varepsilon)$ where M has been previously defined as the maximum of V in $\overline{N}$. Therefore for all $q > 0$

$$qP[\tau_x^\varepsilon > q] \leq \frac{M}{c\varepsilon} .$$

Choosing $q = \frac{2M}{c\varepsilon}$, one obtains $P\{\tau_x^\varepsilon \geq \frac{2M}{c\varepsilon}\} < \frac{1}{2}$. From this fact we have that $P\{K_1^\varepsilon\} \leq \mu_\varepsilon(N)/2$.

Let us now estimate $P(K_3^\varepsilon)$. To do this we first derive the estimate (11) below. Let T' be chosen so that if $x \in S-N$, then $\xi_x^0(t) \in S^c$ for $t > T'$. Note that once the trajectory $\xi_x^0(t)$ leaves $N(Q,S)$ it never returns to $N(Q,S)$. Using (A4), let R' be chosen so that

$$\{\xi_x^0(t) : x \in B(R) \ , \ t > 0\} \subset B(R'-1)$$

and R" be chosen so that

$$\{\xi_x^0(t) : x \in B(R'), \ t > 0\} \subset B(R''-1) \ .$$

Now choose $T > T'$ so that if $x \in B(R') - B(R)$, then $\xi_x^0(t) \in B(R)$ for $t > T'$ and $d(\xi_x^0(t), \partial B(R)) > 2\gamma$ for $t > T'$. This can be done using (A4) and the fact that $B(R') - B(R-1)$ contains no critical points. Now let x be any random initial condition such that $x \in B(R')$ wp1. Then standard estimates yield that

$$P\left\{\sup_{0 \leq t \leq T} |\xi_x^\varepsilon(t) - \xi_x^0(t)| > \gamma\right\} \leq 2n \exp\,(-\gamma\varepsilon^{-1/2}\lambda) \qquad (11)$$

where n is the dimension of the space and λ is a constant depending upon T and a bound on σ, f_x on $B(R'')$. See [1], Lemma 2.1 for a derivation of (11).

Now for sufficiently small ε so that $T < \frac{2M}{c\varepsilon}$,

$$P\{K_3^\varepsilon\} = P\{\tilde{\xi}^\varepsilon(\tfrac{2M}{c\varepsilon}) \in N,\ \tilde{\xi}^\varepsilon(0) \in B(R) - N\}$$

$$\leq \sum_{j=1}^{[\frac{2M}{c\varepsilon T}]} P\left\{\begin{matrix} \tilde{\xi}^\varepsilon(\frac{2M}{c\varepsilon}) \in N,\ \tilde{\xi}^\varepsilon(jT) \notin B(R') - Q, \\ \tilde{\xi}^\varepsilon(kT) \in B(R') - Q,\ \ k = 1,\ldots,j-1. \end{matrix}\right\}$$

$$+ P\{\tilde{\xi}^\varepsilon(\tfrac{2M}{c\varepsilon}) \in N,\ \tilde{\xi}^\varepsilon(kT) \in B(R') - Q,\ k = 1,\ldots,[\tfrac{2M}{c\varepsilon T}]\}.$$

Using estimate (11) appropriately we have that

$$P\{K_3^\varepsilon\} \leq ([\tfrac{2M}{c\varepsilon T}] + 1)2n \exp(-\gamma\varepsilon^{-1/2}\lambda) .$$

Note that $P\{K_3^\varepsilon\} \to 0$ as $\varepsilon \downarrow 0$.

It remains to estimate $P(K_2^\varepsilon)$. The construction of the estimate for $P(K_2^\varepsilon)$ is similar to that for $P(K_3^\varepsilon)$ except that we condition upon the first time that $\tilde{\xi}^\varepsilon(t)$ exits from N. One obtains that $P\{K_3^\varepsilon\} \to 0$ as $\varepsilon \downarrow 0$.

Hence we have

$$\delta = \overline{\lim_{\varepsilon\downarrow 0}}\ \mu_\varepsilon(N) \leq \frac{\delta}{2} + 0 + 0 + \frac{\delta}{6} = \frac{4\delta}{6} .$$

However, this is impossible for $\delta > 0$. Thus δ must be 0 and therefore we can construct a neighborhood of the critical point such that (9) holds.

Now suppose there are k critical points $x_1,\ldots,x_k$ and let N_i be a neighborhood of x_i constructed by the above procedure. Now take any compact set B not intersecting the cycle Γ nor $\bigcup_{i=1}^k N_i$. We claim $\overline{\lim_{\varepsilon\downarrow 0}}\ \mu_\varepsilon(B) = 0$. Suppose not, then $\overline{\lim_{\varepsilon\downarrow 0}}\ \mu_\varepsilon(B) = \delta$ for some $\delta > 0$. Choose R so that $B \subset B(R)$, each $N_i \subset B(R)$ and such that $\mu_\varepsilon(B(R)^c) < \delta/6$. Let ω be such that $d(\Gamma, B) > 2\omega$. Then choose T such that if $x \in B(R) - \bigcup_{i=1}^{k} N_i$, then $d(\xi_x(t), \Gamma) < \omega$ for $t \geq T$. Using the estimate (11) again we have that

$$P\left\{\sup_{0 \leq t \leq T} |\xi_x^\varepsilon(t) - \xi_x^0(t)| > \omega\right\} \to 0 \quad \text{as} \quad \varepsilon \downarrow 0 . \tag{12}$$

Hence

$$\begin{aligned}
\mu^\varepsilon(B) &= P[\tilde{\xi}^\varepsilon(T) \in B] \\
&= P[\tilde{\xi}^\varepsilon(T) \in B,\ \tilde{\xi}^\varepsilon(0) \in B(R) - \bigcup_{i=1}^{k} N_i] \\
&\quad + P[\tilde{\xi}^\varepsilon(T) \in B,\ \tilde{\xi}^\varepsilon(0) \in \bigcup_{i=1}^{k} N_i] \\
&\quad + P[\tilde{\xi}^\varepsilon(T) \in B,\ \tilde{\xi}^\varepsilon(0) \in B(R)^c] .
\end{aligned}$$

Notice that the third term on the right side is bounded above by $\delta/6$ and the second term by $\sum \mu^\varepsilon(N_i)$ which has limit 0 as $\varepsilon \downarrow 0$. Using estimate (12) we have that the first term has limit zero as $\varepsilon \downarrow 0$. Hence

$$\delta = \overline{\lim_{\varepsilon \downarrow 0}}\, \mu_\varepsilon(B) \leq \delta/6$$

contradicting the fact that $\delta > 0$. Therefore $\delta = 0$ and $\lim_{\varepsilon \downarrow 0} \mu_\varepsilon(B) = 0$.

We are now ready to establish (C1). Let B be any open set containing Γ. For any R such that $B \subset B(R)$,

$$\begin{aligned}
1 &= \mu_\varepsilon(B) + \mu_\varepsilon(B(R) - B) + \mu_\varepsilon(B(R)^c) \\
&\leq \mu_\varepsilon(B) + \mu_\varepsilon(B(R) - B) + \delta
\end{aligned}$$

where δ is a bound of $\mu_\varepsilon(B(R)^c)$ for $0 < \varepsilon < \varepsilon_0$. Hence

$1 \leq \varliminf_{\varepsilon \to 0} \mu_\varepsilon(B) + \delta$. By taking R sufficiently large, one obtains that the above inequality holds for δ arbitrarily close to 0, hence $1 = \varliminf_{\varepsilon \downarrow 0} \mu_\varepsilon(B)$ and therefore (C1) holds.

We now prove (C2). Now for any $t > 0$

$$\int H(x) d\mu_\varepsilon(x) = EH(\tilde{\xi}^\varepsilon(t))$$

where, as before, $\tilde{\xi}^\varepsilon(0)$ is distributed according to the measure μ_ε. In particular, then

$$\int_{R^n} H(x) d\mu_\varepsilon(x) = \frac{1}{T^*} \int_0^{T^*} H(\tilde{\xi}^\varepsilon(t)) dt$$

where T^* is the period of the cycle. Since the measure μ_ε becomes concentrated on the cycle Γ as $\varepsilon \downarrow 0$ and H is bounded, we have that

$$\lim_{\varepsilon \downarrow 0} \int H(x) d\mu_\varepsilon(x) = \frac{1}{T^*} \int_0^{T^*} H(\xi_y^0(t)) dt$$

where y is any point on the limit cycle Γ. This completes the proof of Theorem 1. Q.E.D.

Theorem 2 follows immediately from Theorem 1 as discussed in the introduction. Let us now discuss first assumption (A5) and then (A6). Khasminski [8], Wonham [10], Zakai [11], and Kushner [9] have given criteria for the existence of an ergodic measure μ_ε. The results of [8] and [9] depend upon $\sigma\sigma^T$ being uniformly positive definite while the others do not. The remaining condition in (A5) depends upon the construction of an appropriate Liapunov function, see [9] or [10]. A class of problems for which (A5) is satisfied is the following:

(Z) $f(x) = Ax + \sigma g(x)$, σ is constant, g and ∇g are bounded on R^n, A has eigenvalues with negative real parts, (A,B) is controllable and satisfies condition (CO).

For a proof of this fact see the Examples on p. 227 and Corollary 6 on p. 228 in [9]. The definition of condition (CO), a mild restriction in normal applications, is given there. Controllability means that rank $(B, AB, \ldots, A^{n-1}B) = n$.

We now discuss (A6). If $\sigma\sigma^T$ is uniformly positive definite, then conditions guaranteeing the existence of a smooth solution can be found in Theorem 12, p. 25 in [2]. However, it is not necessary that $\sigma\sigma^T$ be positive definite to have a smooth solution, see Theorem 1, p. 73 in [4].

3. In some special cases the invariant measure can be computed explicitly. The left hand side of (2) can then be evaluated using Laplace's method. We illustrate that technique. Let $V(x)$ be a smooth function such that

$$C_1(\varepsilon) = \int_{R^n} \exp(-V(x)/\varepsilon)dx < \infty$$

and let $h(x) = (h_1(x),\ldots,h_n(x))$ satisfy the equations

$$\nabla h \cdot \nabla V = 0 ,$$

$$\operatorname{div} h = 0 .$$

Then the unique invariant probability density associated with the stochastic differential equation

$$d\xi = -\nabla V(\xi) + h(\xi)dt + (2\varepsilon)^{1/2}dw(t) \tag{13}$$

is $p^\varepsilon(x) = C_1(\varepsilon)^{-1} \exp(-V(x)/\varepsilon)$. To prove this one simply checks that p^ε satisfies the Fokker-Planck equation.

Let $x \in R^2$, $V(x) = (1 - x_1^2 - x_2^4)^2$, $h_1(x) = 4x_2^3$ and $h_2(x) = -2x_1$. Then $\Gamma = \{x : x_1^2 + x_2^4 = 1\}$ is a stable limit cycle whose domain of attraction is $R^2 - \{0\}$.

Let us evaluate $D(\varepsilon) = \int_{R^2} x_2^2 d\mu_\varepsilon(x)$. Thus

$$D(\varepsilon) = \frac{\int_{R^2} x_2^2 \exp(-V(x)/\varepsilon)dx}{\int_{R^2} 1 \exp(-V(x)/\varepsilon)dx} .$$

Making the change of variables $s = x_1^2 + x_2^4$, $t = x_1$, and using symmetry, one obtains that

$$D(\varepsilon) = \frac{\int_0^\infty \int_0^{\sqrt{s}} R_1(s,t)\ \exp\ (-(1-s)^2/2\varepsilon)dtds}{\int_0^\infty \int_0^{\sqrt{s}} R_2(s,t)\ \exp\ (-(1-s)^2/2\varepsilon)dtds}$$

where $R_1(s,t) = 4^{-1}(s-t^2)^{-1/4}$ and $R_2(s,t) = 4^{-1}(s-t^2)^{-3/4}$.

Define $M_1(s) = \int_0^{\sqrt{s}} R_1(s,t)dt$ and $M_2(s) = \int_0^{\sqrt{s}} R_2(s,t)dt$. Utilizing Laplace's method one obtains that

$$\lim_{\varepsilon\downarrow 0} D(\varepsilon) = \frac{M_1(1)}{M_2(1)} = \frac{\int_0^1 (1-t^2)^{-1/4}dt}{\int_0^1 (1-t^2)^{-3/4}dt}\ . \tag{14}$$

Now let $\xi_1(t)$, $\xi_2(t)$ be the solution to (13) with $\varepsilon = 0$ and $\xi_1(0) = 0$, $\xi_2(0) = 1$. Since $\xi_1^2(t) + \xi_2^4(t) = 1$, then the right hand side of (2) is

$$\frac{4}{T^*} \int_0^{T^*/4} \sqrt{1 - \xi_1^2(t)}\ dt\ .$$

Making the change of variables $s = \xi_1(t)$, one again obtains (14). The conclusion of Theorem 1 holds for this example even though assumption (A2) is not satisfied. Note that the matrix $f_x(0)$ has 0, 2 as eigenvalues.

4. In this section we consider a new case where the origin is the unique critical point of (1) with $\varepsilon = 0$ and is globally asymptotically stable. It is then easy to establish Theorem 3 using a method similar to that employed in the proof of Theorem 1. The details are omitted.

However, in [6] under stronger assumptions than in Theorem 3 we established an expansion

$$\int_{R^n} H(x)d\mu_\varepsilon(x) = c_0 + \sum_{j=1}^{k} c_j\varepsilon^j + o(\varepsilon^k) \tag{15}$$

valid for any positive integer k. The constants c_j can be found by solving linear algebraic equations with $c_0 = H(0)$. Among the assumptions in [6] are that the matrix $f_x(0)$ have only eigenvalues with negative real parts. See [6] or [7] for details.

The expansion (15) is not always to be expected. Consider $d\xi = -\xi^3 dt + (2\varepsilon)^{1/2}dw(t)$. This equation has an invariant measure for $\varepsilon > 0$ given by $B(\varepsilon)\exp(-x^4/4\varepsilon)$ where $B(\varepsilon)$ is a normalizing constant. An easy calculation shows that

$$\int_R x^2 d\mu_\varepsilon(x) = C\varepsilon^{1/2}$$

for an appropriate constant C. This contradicts the expansion (15). The difficulty is that $f_x(0) = 0$ so that the rate of approach to the origin is too slow.

One might be tempted to derive an expansion of (2) in powers of ε. However, one will probably need an additional assumption guaranteeing that the rate of approach of the deterministic trajectories to Γ near Γ is "sufficiently fast".

References

1. W. Fleming, Stochastically perturbed dynamical systems, Rocky Mountain J. of Math. 4 (1974), 407-433.

2. A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, 1964.

3. A. Friedman, Stochastic Differential Equations and Applications, vol. 1,2, Academic Press, New York, 1976.

4. I. Gihman and A. Skorokhod, Stochastic Differential Equations, Springer-Verlag, New York, 1972.

5. J. Hale, Ordinary Differential Equations, Wiley-Interscience, New York, 1969.

6. C. Holland, Ergodic expansions in small noise problems, J. Differential Eqns. 16 (1974), 281-288.

7. C. Holland, Stationary small noise problems, Internat. J. Nonlinear Mech. 11 (1976), 41-47.

8. R. Khasminski, Ergodic properties of recurrent diffusion processes and stabilization of the solution to the Cauchy problem for parabolic equations, Theor. Prob. Appl. 5 (1960), 179-196.

9. H. Kushner, The Cauchy problem for a class of degenerate parabolic equations and asymptotic properties of the related diffusion processes, J. Differential Equations 6 (1969), 209-231.

10. W. Wonham, A Lyapunov method for the estimation of statistical averages, J. Differential Equations 2 (1966), 365-377.

11. M. Zakai, A Lyapunov criterion for the existence of stationary probability distributions for systems perturbed by noise, SIAM J. Control 7 (1969), 390-397.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE | READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 19 18 AFOSR TR-77-1276	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) 6 STOCHASTICALLY PERTURBED LIMIT CYCLES,		5. TYPE OF REPORT & PERIOD COVERED ~~Interim~~
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 Charles J. Holland		8. CONTRACT OR GRANT NUMBER(s) 15 AFOSR-77-3296
9. PERFORMING ORGANIZATION NAME AND ADDRESS Purdue University Department of Mathematics West Lafayette, Indiana 47907		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 16 2304 17 A1
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB DC 20336		12. REPORT DATE 11 Aug 1977
		13. NUMBER OF PAGES ~~14~~ 12 20p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An important question in the stability and control of stochastic systems is the determination of the limiting long time behavior of the system using a fixed control. This paper answers that question in the case of a stochastic system per- → next page

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

402451 SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. ABSTRACT (Continued)

turbed by a small additive noise term where the control is such that the corresponding deterministic system has a stable limit cycle.

It is shown that in the limit of large time the stochastic system is near the limit cycle. This is a stability result. Moreover, one can compute approximately at which portion of the limit cycle one is most likely to be found. Further various stationary averages can be computed.

These results will be of use in designing approximate controls for stationary stochastic control systems. As a by-product, the above work allows one to deduce some additional results on singular perturbation problems in partial differential equations.